Understanding Problems to Calculate Non-Trivial Homotopy Groups of Spheres

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Abstract— One of the fundamental problems of Algebraic topology is to calculate Homotopy groups of spheres that is concerned as a goal of 21st century. Trivial homotopy groups of spheres have been calculated convincingly. But computation of non-trivial homotopy groups of spheres is found very complex and difficult. To understand the complexity, we have reviewed computation of the trivial homotopy groups and then looked at the complexity again to draw some meaningful insights. We found that homotopy groups have not been defined properly yet for non-trivial cases and have no useful classification of it. Some proofs have been rewritten to ensure their mathematical foundations.

Keyword— Fundamental group, Homotopy group, Covering space, Lifting properties, Cell complex, Cellular approximation, Homotopy extension property.

1 Introduction

One of the fundamental problems in algebraic topology [19] and the ultimate goal of 21st century is to calculate homotopy group of sphere [8]. Computation of non-trivial homotopy groups of k-spheres is complex and difficult [2, 7, 14], so it could not be studied in a general case study [7]. Also, new mathematics has been generated concerning the computation of non-trivial homotopy groups which is another motivation to this investigation.

There are some studies have been done concerning homotopy groups of spheres. After some ground breaking works of computing homotopy groups of sphere, such as [11, 13, 22], development of this field of study concentrated on calculations of non-trivial homotopy groups of spheres. Most of the studies have been conducted concerning the stable homotopy groups of spheres [6, 9, 18, 23].

Homotopy groups have been reinvestigated for $i \geq 2$ in [21] by giving alternative proof of Gray's results. Mark Mahowald's contribution to compute homotopy groups of spheres have been reviewed in [12]. Besides these review works, there are very few recent literatures on calculating homotopy groups of spheres. Some unpublished lecture notes [2, 3, 15, 17, 20] have been found on this topic. A thesis work [10] has been performed recently proofing Pontrygin's theorem. Homotopy group of spheres have been recently investigated concerning Hopf fibration and Villarceau circles in [5].

In this article, we have reviewed calculation of homotopy groups of k-spheres with a goal to understand problems of computing non-trivial homotopy groups of k- spheres. Since the complexity of calculating non-trivial homotopy group of spheres has not removed yet, we need to understand existing knowledge of it in depth.

2 Background and Methodology

2.1 Fundamental group: A map $f : X \to Y$ said to be homotopic to a map $g : X \to Y$ if there exists a map $F: X \times I \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for $x \in X$. The map F is called a homotopy deforms f to g. It is denoted by $F: f \simeq g$. If f and g are two paths in X having the same initial point x_0 and the final point x_1, then the homotopy F deforms f to g together with two conditions $F(0,t) = x_0$ and $F(0,t) = x_1$ for $t \in I$. In that case, F is called a path homotopy. The relation "$\simeq$" is an equivalence relation. The homotopy class of f is denoted by $[f]$ is the equivalence class under the relation $\simeq$. Let f be a path in X from x_0 to x_1, and let g be a path in X from x_1 to x_2. The product $f * g$ is also a path from x_0 to x_2 defined by

$$(f * g)(s) = \begin{cases} f(2s) & for\ \ s \in [0, ½] \\ g(2s-1) & for\ \ s \in [½, 1] \end{cases}$$

This product gives an operation defined by $[f] * [g] = [f * g]$ in the set of path homotopy classes. This operation satisfies the properties, which is similar to the axioms for a group. One difference from the properties of a group is that $[f] * [g]$ is not defined for every pair of classes, but only for those pairs $[f], [g]$ for which $f(1) = g(0)$. The following theorem shown that this operation satisfies associativity, identity and inverse law.

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2.2 Theorem:
Let f, g and h be three paths in a space X with $f(0) = x_0$, $f(1) = x_1 = g(0)$ and $g(1) = x_2 = h(0)$, then

(1) Associativity: $[f] * ([g] * [h])= ([f] * [g]) * [h]$.

(2) Identity: If $e_x: I \to X$ denote the constant path in X at $x \in X$. Then $[f] * [e_{x_1}] = [f]$ and $[e_{x_0}] * [f] = [f]$.

(3) Inverse: Let $\bar{f}(s) = f(1 - s)$ for all $s \in X$. Then $\bar{f}$ also a path in X from x_1 to x_0 with $[f] * [\bar{f}] = [e_{x_0}]$ and $[\bar{f}] * [f] = [e_{x_1}]$. Here $\bar{f}$ is called the reverse of f.

Proof: To prove the problem, we shall use the following two conditions.

i) If $k: X \to Y$ is a continuous map, and if F is a path homotopy in X between the paths f and f', then $k \circ F$ is a path homotopy in Y between the paths $k \circ f$ and $k \circ f'$.

ii) if $k: X \to Y$ is a continuous map and if f and g are path in X with $f(1) = g(0)$. Then

$$k \circ (f * g) = (k \circ f) * (k \circ g).$$

First, we prove **(2)**. Let e_0 be the constant path in I at 0, and let i be the identity map in I, which is a path in I. then the composition $e_0 * i$ is also a path in I from 0 to 1.
Since I is convex, there exists a path homotopy G between i and $e_0 * i$. From condition (i), $f \circ G$ is a path homotopy between the paths $f \circ i$ and $f \circ (e_0 * i)$. But $f \circ i = f$ and from the condition (ii), $f \circ (e_0 * i) = (f \circ e_0) * (f \circ i) = e_{x_0} * i$.
Therefore, $[e_{x_0}] * [f] = [f]$. Similarly, $[f] * [e_{x_1}] = [f]$.
To prove **(3)**, Let $\bar{\iota} : I \to I$ be the reverse of i.
i.e., $\bar{\iota}(t)$=1-t, which is a path in I from 1 to 0. Then the product $i * \bar{\iota}$ is a path the in I that begins and ends at 0.
Since I is convex, there exists a path homotopy H between the paths $i * \bar{\iota}$ and e_0. Then $f \circ H$ is a path homotopy between the paths $f \circ e_0$ =e_{x_0} and $\circ (i * \bar{\iota}) = (f \circ i) * (f \circ \bar{\iota}) = f * \bar{f}$. Thus, $[f] * [\bar{f}] = [e_{x_0}]$. Similarly, we can prove that $[\bar{f}] * [f] = [e_{x_1}]$.
To prove **(1)**, first we define the product $f * g$ in another way.
Let $p: [a, b] \to [c, d]$ be a map defined by $p(x) = mx + c$, where $m, c \in \mathbb{R}$ be such that $p(a) = c$ and $p(b) = d$. The map p is continuous, it called the positive linear map from the interval $[a, b]$ onto the interval $[c, d]$, such map is unique. The inverse of p is also a positive linear map, and so is the composite of two such maps.
If p is the positive linear map from [0, ½] onto [0,1] and if q is the positive linear map from [½, 1] onto [0,1], then define

$$(f * g)a(s) = \begin{cases} f \circ p & for \ \ s \in [0, ½] \\ g \circ q & for \ \ s \in [½, 1] \end{cases} \quad \text{for } \ s \in [0,1]$$

Since the composite of two continuous maps is also continuous, $f \circ p$ and $g \circ q$ are continuous. The continuity of $f * g$ comes from the pasting lemma.
Now we prove **(1)**. Choose $a, b \in [0,1]$ such that $0 < a < b < 1$. If $p, q \ and \, r$ are the positive linear maps defined from the different domains $[0, a]$, $[a, b]$ and $[b, 1]$ respectively onto [0,1]. We define $k_{a,b}: I \to X$ by

$$k_{a,b}(s) = \begin{cases} f \circ p & for \ \ s \in [0, a] \\ g \circ q & for \ \ s \in [a, b] \\ h \circ r & for \ \ s \in [b, 1] \end{cases} \text{for all } s \in [0,1].$$

Here $k_{a,b}$ is continuous, and depend on the choice of a, b. Therefore, if we choose another pair $c, d \in [0,1]$ with $0 < c < d < 1$, then $k_{a,b}$ may not be equal to $k_{c,d}$. But we claim that they are homotopic.
Let u be the path in I from $0 \ to \ 1$, made up by joining of the positive linear maps from the domains $[0, a], [a, b]$ and $[b, 1]$ onto $[0, c], [c, d]$ and $[d, 1]$ respectively. Indeed, $k_{c,d} \circ u = k_{a,b}$. Since I is convex, u is homotopic to i, the identity map in I. If $U$is a path homotopy between u and i, then $k_{c,d} \circ U$ is a path homotopy between $k_{c,d} \circ u = k_{a,b}$ and $k_{c,d} \circ i = k_{c,d}$. (by using condition (i)). Thus $[k_{a,b}] = [k_{c,d}]$. Finally, if we put $a = 1/2$, and $b = 3/4$, we see that $k_{a,b}$ is equal to the product $f * (g * h)$, and if we put $c = 1/4 \ and \ d = 1/2$, we see that $k_{c,d}$ is equal to the product $(f * g) * h$. This completes the proof.
The set of path homotopy classes of X don't form a group under the operation $*$. To form a group choose a point x_0 in X, the set of all path-homotopy classes of the paths that begin and end at x_0 form a group under $*$. It is called the fundamental group of X relative to the base point x_0, it is denoted by $\pi_1(X, x_0)$. A path-connected space$X$ is said to be simply connected if $\pi_1(X, x_0)$ is trivial for each $x_0 \in X$.

2.3 Covering spaces:
Let $p: E \to B$ be a continuous and surjective map. An open set U of B is called evenly covered by p if

$$p^{-1}(U) = \bigcup_{\alpha} V_\alpha$$

where V_α, s are the disjoint open sets in E, and ${}^{p}/_{V_\alpha} : V_\alpha \to U$ is an isomorphism for each α. The collection $\{V_\alpha\}$ is called a partition of $p^{-1}(U)$ into slices. Note that if U is evenly covered by p then every subsets of U is evenly covered by p. If for every $b \in B$ there is a neighborhood U of b which is evenly covered by p, then p is said to be a covering map, and the space E is called a covering space of B.

2.4 Example. The map $p: \mathbb{R} \to S^1$ be define by

$$p(x) = (cos2\pi x, sin2\pi x)$$

is a covering map.

2.5 Lifting Properties:
Let $p: E \to B$ be a map. If f is a continuous map from a space X into B, a lifting of f is a map $\bar{f}: X \to E$ such that $p \circ \bar{f} = f$.
Let $p: E \to B$ be a covering map. Choose e_0 for which $p(e_0) = b_0$. Then the map $\varphi: \pi_1(B, b_0) \to p^{-1}(b_0)$ definied by

$$\varphi([f]) = \bar{f}(1) \ \text{ for all } [f] \in \pi_1(B, b_0)$$

is called the lifting correspondence induced by the covering map p. Where $\bar{f}$ is the unique lifting of f, which is a path in E with $\bar{f}(0) = e_0$. Since $\bar{f}$ is unique, φ is well-defined. The correspondence φ depends on the choice of e_0.

2.6 Theorem:
If $p: E \to B$ is a covering map with $p(e_0) = b_0$, and if E is a path connected space, then the lifting correspondence

$$\varphi: \pi_1(B, b_0) \to p^{-1}(b_0)$$

is surjective. If E is a simply connected space, then it is bijective [16].

2.7 Deformation retraction:
A deformation retraction of a space X onto a subspace A is a family of maps $f_t: X \to X$, $t \in I$, such that $f_0 = i$ (the identity map), $f_1(X) = A$, and $f_t/A = i$ for all t. The family $f_t$should be

continuous in the sense that the associated map $X \times I \to X$, $(x,t) \to f_t(x)$ is continuous. We say that A is a deformationretract of X if there exists a deformation retraction of X onto A.

A deformation retraction $f_t: X \to X$ is a special case of the general notion of ahomotopy, which is simply any family of maps $f_t: X \to X$, $t \in I$, such that the associated map $F: X \times I \to Y$given by $F(x,t) = f_t(x)$ is continuous.

2.8 Relative homotopy:

A homotopy $f_t: X \to X$ that gives a deformation retraction of Xonto a subspace A has the property that $f_t/A = i$ for all t . In general, a homotopy $f_t: X \to Y$ whose restriction to a subspace A of X is independent of t is called a homotopy relative to A, or more concisely, a homotopy rel A. Thus, a deformation retraction of Xonto A is a homotopy rel$A$from the identity map of X to a retraction of X onto A.

2.9 Homotopy equivalent:

If a space X deformation retracts onto a subspace A via $f_t: X \to X$, then if $r: X \to A$ denotes the resulting retraction and $j: A \to X$ the inclusion, we have $rj \simeq i$ and $jr \simeq i$, the latter homotopy being given by f_t. Generalizing this situation, a map $f: X \to Y$ is called a homotopy equivalence if there is a map $g: Y \to X$ such that fg and gf are the identity maps of X and Y respectively. The spaces X and $Y$are said to be homotopy equivalent or to have the same homotopy type.

2.10 Contractible space:

A space is contractible if it is homotopy equivalent to a one-point space.

2.11 Homotopy Extension property:

Given a map $f: X \to Y$. A subspace A of X is said to have the homotopy extension property (HEP) in X with respect to a space Y, if every homotopy

$$f_t: A \to Y (0 \le t \le 1),$$

of the map f/A has an extension

$$g_t: X \to Y (0 \le t \le 1),$$

such that $g_0 = f$, therefore g_t is a homotopy of f. We also say that the pair (X, A) satisfies the homotopy extension property.

2.12 Homotopy Group:

Let X be a topological space with a base point x_0. For $n \ge 1$ the nth homotopy group $\pi_n(X, x_0)$ of X is defined to be the homotopy classes of maps from the n-cube I^n to X, which sends the faces ∂I^n to x_0. Thus by an element of $\pi_n(X, x_0)$ we mean a homotopy class of maps

$$\alpha: (I^n, \partial I^n) \to (X, x_0)$$

Equivalently, an element of $\pi_n(X, x_0)$ is a homotopy class of maps $(S^n, p) \to (X, x_0)$ for some base point p.

Two maps $\alpha, \beta: (I^n, \partial I^n) \to (X, x_0)$ represent the same element of $\pi_n(X, x_0)$ if and only if there is a homotopy $H: I^n \times I \to X$ such that

$$H(-,0) = \alpha,\ H(-,1) = \beta, \text{and } H(\partial I^n, I) = x_0.$$

2.13 Proposition:

For each pointed space (X, x_0) and $n \ge 1$, the set $\pi_n(X, x_0)$ is a group, the nth homotopy group of (X, x_0).

Proof: If α and β are two maps from I^n to X, representing $[\alpha]$ and $[\beta]$ in $\pi_n(X, x_0)$, then the product $[\alpha] * [\beta]$ is the homotopy class of the map

$$(\alpha * \beta)(t_1, t_2, \dots, t_n) = \begin{cases} \alpha(2t_1, t_2, \dots, t_n) & \text{for } 0 \le t_1 \le ½ \\ \beta(2t_1 - 1, t_2, \dots, t_n) & \text{for } ½ \le t_1 \le 1 \end{cases}$$

Notice that the definition agrees with the known group structure on the fundamental group for $n = 1$. The proof that the group operation $*$ is well defined, associative, that the constant map $e_{x_0}: I^n \to X$ represents the identity element, and that each element $[\alpha]$ has an inverse represented by

$$\alpha^{-1}(t_1, t_2, \dots, t_n) = \alpha(1 - t_1, t_2, \dots, t_n)$$

is exactly the same as for $\pi_1(X, x_0)$, and we leave the details.

One may object that the definition of the group structure is a bit unnatural, because the first coordinate t_1 is given a preferred role in the definition of the group structure. We could also define a product as follows:

$$(\alpha *_i \beta)(t_1, t_2, \dots, t_n) = \begin{cases} \alpha(t_1, \dots, 2t_i, \dots, t_n) & \text{for } 0 \le t_i \le ½ \\ \beta(t_1, \dots, 2t_i - 1, \dots, t_n) & \text{for } ½ \le t_i \le 1 \end{cases}$$

The explanation is that these two products induce the same operation on homotopy classes. The proof of this fact is given by the following observation (*lemma* 2.14) together with the so-called Eckmann-Hilton argument (*proposition* 2.15).

2.14 Lemma:

The operation $*$ distributes over the operation $*_i$ in the sense that

$$(\alpha *_i \beta) * (\gamma *_i \delta) = (\alpha * \gamma) *_i (\beta * \delta).$$

Proof: We only have to look at the case $n = 2; i = 2$. Then the expressions on the left and right correspond to the same subdivisions of the square so define identical maps.

2.15 Proposition: $('Eckmann - Hilton\ trick')$

Let S be a set with two associative operations $*, \circ: S \times S \to S$ having a common unit $e \in S$. Suppose $*$ and $\circ$ distribute over each other, in the sense that

$$(\alpha * \beta) \circ (\gamma * \delta) = (\alpha \circ \gamma) * (\beta \circ \delta)$$

Then $*$ and $\circ$ coincide, and define a commutative operation on S [4].

Proof: Taking $\beta = e = \gamma$ in the distributive law yields $\alpha \circ \delta = \alpha * \delta$

Applying this proposition to $*$ and $*_i$ shows that these define the same operation on $\pi_n(X, x_0)$ for $n \ge 2$. The proposition also shows:

2.16 Corollary:

The groups $\pi_n(X, x_0)$ are abelian for $n \ge 2$.

2.17 Proposition:

For each map $(X, x_0) \to (Y, y_0)$ the induced operation $\pi_n(X, x_0) \to \pi_n(Y, y_0)$ is a group homomorphism, defining a functor π_n: $\boldsymbol{Top_*} \to \boldsymbol{Grp_*}$. The functor π_n is homotopy invariant. That is, $\pi_n(\alpha) = \pi_n(\beta)$ for homotopic maps $\alpha \simeq \beta$.

Proof: Given a map $f: (X, x_0) \to (Y, y_0)$. Let α and β be two maps from I^n to X, representing $[\alpha]$ and $[\beta]$ in $\pi_n(X, x_0)$. The map f induces a map

$$f_*: \pi_n(X, x_0) \to \pi_n(Y, y_0),$$
$$f \to f \circ \alpha.$$

This makes π_n a functor from pointed topological spaces to groups. Moreover, if H is a homotopy between α and β, then clearly $f \circ H$ is a homotopy between $f \circ \alpha$ and $f \circ \beta$. Thus, the functor π_n is homotopy invariant. Again $(f \circ \alpha) * (f \circ \beta) = f \circ (\alpha * \beta)$ implise f_* is a homomorphism.

2.18 CW complex:

Let X and Y be two topological spaces, and let A be a subspace of Y. If $f: A \to Y$ is a continuous map, then the adjunction space (or attaching space) $X \cup_f Y$ of X with Y along A with via f is the quotient space of the disjoint union of X and Y under the identification $a \sim f(a)$ for all $a \in A$. Here the map f is called an attaching map.

A CW complex (cell complex) is a space X constructed in the following way:

(1) Beginning with a set X^0 of points, each point isconsidered as 0-cell.

(2) Inductively, form the n-skeleton X^n from X^{n-1} by attaching n −cells e_α^n via maps $\varphi_\alpha: S^{n-1} \to X^{n-1}$. This means that X^n is the quotient space of the disjoint union $X^{n-1} \sqcup_\alpha D_\alpha^n$ of X^{n-1} with a collection of n-disks D_α^n under the identifications $x \sim \varphi_\alpha(x)$ for $x \in \partial D_\alpha^n$. Thus as a set, $X^n = X^{n-1} \sqcup_\alpha e_\alpha^n$ where each $e_\alpha^n = \{x \in \mathbb{R}^n : \|x\| < 1\}$, an open n-disk.

(3) One can either stop this inductive process at a finite stage, setting $X = X^n$ for some $n < \infty$, or one can continue indefinitely, setting$X = \cup_n X^n$. In the latter case X is given the weak topology: A set $A \subset X$ is open (or closed) if and only if $A \cap X^n$ is open (or closed) in X^n for each n.

If $X = X^n$ for some n, then X is said to be finite-dimensional, and the smallest such n is the dimension of X, the maximum dimension of cells of X.

2.19 Example:

The sphere S^n has the structure of a cell complex with just two cells, e_0 and e_n, the n cell being attached by the constant map $S^{n-1} \to e_0$. This is equivalentto regarding S^n as the quotient space $D^n/\partial D^n$.

Let X be a cell complex. Then for each cell e_α^n in X there is an extension $\phi_\alpha: D_\alpha^n \to X$ of the attaching map φ_α for which the interior of D_α^n is homeomorphic to e_α^n, called the characteristic map of e_α^n. We can take ϕ_α to be the composition $D_\alpha^n \hookrightarrow X^{n-1} \sqcup_\alpha D_\alpha^n \to X^n \hookrightarrow X$ where the middle map is the quotient map defining X^n. Being a composition of continuous map, ϕ_α is continuous.

A sub complex of a cell complex A is a closed subspace $A \subset X$ that is a union of cells of X. Since A is closed, the characteristic map of each cell in A has image contained in A, and in particular the image of the attaching map of each cell in A is contained in A, so A is a cell complex in its own right. A pair (X, A) consisting of a cell complex X subcomplex A will be called a CW pair.

In a weak topology of a cell complex, a set is closed if and only if it meets the closure of each cell in a closed set. For if a set meets the closure of each cell in a closed set, it pulls back to a closed set under each characteristic map, hence is closed.

2.20 Proposition:

If (X, A) is a CW pair, then $X \times \{0\} \cup A \times I$ is a deformation retract of $X \times I$, hence (X, A) has the homotopy extension property.

2.21 Suspension:

Let X be a space, the quotient space $X \times 1$ is said to be a suspension of X if the points $X \times \{0\}$ and $X \times \{1\}$ collapses to two different points. A suspension can be considered as a double cones $CX = (X \times 1)/(X \times \{0\})$. For a map $f: X \to Y$, the suspension map $Sf: SX \to SY$, is the quotient map $f \times i_d: X \times I \to Y \times I$, where i_d is the identity map of $I = [0,1]$.

3 RESULTS AND DISCUSSION

3.1 Fundamental group of 1 −sphere, $\pi_1(S^1) \cong \mathbb{Z}$

3.1.1Theorem:

The fundamental group of S^1 is an infinite cyclic group.

Proof: To show that the fundamental group of S^1 is an infinite cyclic group we shall show that it is isomorphic to the additive group of integers.

Consider the covering map $p: \mathbb{R} \to S^1$ defined by

$$p(x) = (\cos 2\pi x, \sin 2\pi x).$$

let $b_0 = (1, 0)$. Then the inverse image $p^{-1}(b_0)$ is the set $\mathbb{Z}$ of all integers. Because $\mathbb{R}$ is simply connected, the lifting correspondence $\varphi: \pi_1(S^1, b_0) \to \mathbb{Z}$ is bijective. We have to prove that φ is a homomorphism.

Let $[f], [g] \in \pi_1(S^1, b_0)$. Let $\bar{f}$ *and* $\bar{g}$ be the liftings of f *and* g respectively with the same initial point 0. If $\bar{f}(1) = m$ and $\bar{g}(1) = n$, then $\varphi([f]) = m$ and $\varphi([g]) = n$. Define

$$\bar{\bar{g}}(s) = m + \bar{g}(s).$$

Then $\bar{\bar{g}}$ is a path in $\mathbb{R}$ with $\bar{\bar{g}}(0) = n$. Since for all $x \in \mathbb{R} p(m + x) = p(x)$, $\bar{\bar{g}}$ is a lifting of g. The product $\bar{f} * \bar{\bar{g}}$ is also a path that begins at 0 and ends at $\bar{\bar{g}}(1) = m + \bar{g}(1) = m + n$, and it is the lifting of $f * g$. Then

$$\begin{aligned} \varphi([f] * [g]) &= \varphi([f * g]) \\ &= (\bar{f} * \bar{\bar{g}})(1) \\ &= m + n \\ &= \varphi([f]) + \varphi([g]). \end{aligned}$$

Thus, the proof is completed.

3.2 Homotopy groups $\pi_i(S^1)$ for $i > 1$

3.2.1 Theorem:

For $i > 1$, $\pi_i(S^1) \cong 0$.

Proof: For $n > 1$, $\mathbf{S}^n$ is simply connected, therefore $\pi_1(S^n) \cong 0$. So any map $f: S^n \to S^1$ induces a map $f_*: \pi_1(S^n) \to \pi_1(S^1)$ which is equivalent to the unique map $0 \to \mathbb{Z}$. The image of f_* lies in the induced map of the covering map $p: \mathbb{R} \to S^1$ as $\mathbb{R}$ is the universal cover of S^1 and $\pi_1(\mathbb{R})$ is trivial. If $\bar{f}: S^n \to \mathbb{R}$ is the lifting of f then $\bar{f}$ is nullhomotopic. Since $\mathbb{R}$ is contractible, f is also nullhomotopic.

3.3 Homotopy groups $\pi_i(S^n)$ for $i < n$

3.3.1 Cellular Approximation

To prove $\pi_i(S^n) = 0$ for $i < n$, we need to prove that every map $S^i \to S^n$ can be deformed in such a way that its image miss at least one point of S^n, and with the fact that the complement of a point in S^n is contractible to end the proof. First step ensures that there is no continuous map $S^i \to S^n$ could be surjective when $i < n$, but space-filling curves from point-set topology can be used to produce such maps. To validate this strategy, homotopies can be constructed removing these dimension-raising maps.

3.3.2 Definition:

If X and Y are CW complexes, and $f: X \to Y$ is a continuous map, then f is said to be cellular, if it takes the $m -$ skeleton of X to the $n -$ skeleton of Y for all m and $m \geq n$.

3.3.3 Theorem: (Cellular approximation theorem [1])

Every map $f: X \to Y$ of CW complexes is homotopic to a cellular map. The homotopy can be taken to be stationary on A when f is already cellular on a sub complex $A \subset X$.

3.3.4 Corollary:

$\pi_i(S^n) = 0$ for $i < n$.

Proof:

Let usual CW structures of S^i and S^n with the $0 -$cells as basepoints are given, then homotopy can be found for each basepoint-preserving map $S^i \to S^n$ which can be considered to be cellular fixing the basepoints. Therefore the map is constant for $i < n$.

3.4 Homotopy groups $\pi_i(S^n)$ for $i = n$

3.4.1 Theorem: (Excision for Homotopy Groups)

Suppose X is a CW complex, A and B be two sub complexes such that $A \cup B = X$ and $A \cap B = C$ is non-empty. If (A, C) is $m -$ connected and (B, C) is $n -$ connected, then the inclusion map $(A, C) \to (B, C)$ induces an isomorphism for $i < m + n$ and a epimorphism for $i = m + n$.

3.4.2 Theorem:(Freudental Suspension Theorem)

If X is a $(n - 1)$ connected CW complex then the suspension map $\pi_i(X) \to \pi_i(SX)$ is an isomorphism.

Proof: From the definition of suspension one can considered that SX as a double cones C_0X and C_1X on X such that their union is SX and intersection is $X \times \{\frac{1}{2}\}$. From the long exact sequence of pair there are isomorphisms $\pi_i(X) \to \pi_{i+1}(C_1X, X)$ and $\pi_{i+1}(X) \to \pi_{i+1}(SX, C_0X)$ and (C_1X, X) is $n -$connected. By the preceding theorem the inclusion map induces an isomorphism $\pi_{i+1}(C_1X, X) \cong \pi_{i+1}(SX, C_0X)$ composing these three isomorphism we have, $\pi_i(X) \cong \pi_{i+1}(SX)$.

3.4.3 Corollary: $\pi_n(S^n) \cong \mathbb{Z}$

Proof: From the previous corollary we have $\pi_1(S^1) \to \pi_2(S^2) \cong \pi_3(S^3) \cong \cdots\cdots$, the first map is surjective since $\pi_1(S^1) \cong \mathbb{Z}$, $\pi_2(S^2)$ is a cyclic group. The long exact sequence of Homotopy groups of the Hopf bundle gives $\pi_1(S^1) \cong \pi_2(S^2)$ and hence $\pi_n(S^n) \cong \mathbb{Z}$.

3.5 Homotopy groups $\pi_i(S^n)$ for $i > n$

Computation methods of the groups $\pi_i(S^n)$ for $i > n$ are very difficult and also most of the groups of this type are nontrivial. This difficulty comes from the following senses:

- Homotopy group can not be defined constructively for $i > n$.
- We can not make any comment using homology groups too, since $H_i(S^n) \cong 0$ for $i > n$.
- Homotopy groups $\pi_i(S^n)$ for $i > n$ can not be classified into some specific classes like as $\pi_i(S^n)$for $i \leq n$.

6 Conclusion

Computation of homotopy groups of spheres has been reviewed with some understandings. From these understandings few problems have been mentioned. This study can be extended to determine stable homotopy groups of spheres similar to [6], in the application of homotopy groups to other fields of knowledge and to find new techniques to compute non-trivial homotopy groups of sphere/non-countable spaces.

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